

BUBBLE CONCENTRATION ON SPHERES FOR SUPERCRITICAL ELLIPTIC PROBLEMS

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ABSTRACT. We consider the supercritical Lane-Emden problem

$$(P_\varepsilon) \quad -\Delta v = |v|^{p_\varepsilon-1}v \text{ in } \mathcal{A}, \quad u = 0 \text{ on } \partial\mathcal{A}$$

where \mathcal{A} is an annulus in \mathbb{R}^{2m} , $m \geq 2$ and $p_\varepsilon = \frac{(m+1)+2}{(m+1)-2} - \varepsilon$, $\varepsilon > 0$.

We prove the existence of positive and sign changing solutions of (P_ε) concentrating and blowing-up, as $\varepsilon \rightarrow 0$, on $(m-1)$ -dimensional spheres. Using a reduction method ([18, 14]) we transform problem (P_ε) into a nonhomogeneous problem in an annulus $\mathcal{D} \subset \mathbb{R}^{m+1}$ which can be solved by a Ljapunov-Schmidt finite dimensional reduction.

1. INTRODUCTION

In this paper we address the question of finding solutions concentrated on manifolds of positive dimension of supercritical elliptic problems of the type

$$-\Delta v = |v|^{p-1}v \text{ in } \mathcal{A}, \quad u = 0 \text{ on } \partial\mathcal{A}, \quad (1)$$

where $\mathcal{A} := \{y \in \mathbb{R}^d : a < |y| < b\}$, $a > 0$, is an annulus in \mathbb{R}^d , $d > 2$ and $p > \frac{d+2}{d-2}$ is a supercritical exponent.

We remark that the critical and supercritical Lane-Emden problems are very delicate due to topological and geometrical obstruction enlightened by the Pohozaev's identity ([16]). We also point out that in the supercritical case a result of Bahri-Coron type ([2]) cannot hold in general nontrivially topological domains as shown by a nonexistence result of Passaseo ([15]), obtained exploiting critical exponents in lower dimensions. Using similar ideas, some results for exponents p which are subcritical in dimension $n < d$ and instead supercritical in dimension d have been obtained in different kind of domains in [1, 4, 6, 8, 9, 10, 11, 13].

Here we consider annuli in even dimension $d = 2m$, $m \geq 2$ and obtain results about the existence of solutions, both positive and sign changing, of different type, concentrated on $(m-1)$ -dimensional spheres. More precisely, we have

Theorem 1.1. *[Case of positive solutions] Let $\mathcal{A} \subset \mathbb{R}^{2m}$, $m \geq 2$ and define $(\partial\mathcal{A})_a := \{y \in \partial\mathcal{A} : |y| = a\}$. There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the following supercritical problem*

$$-\Delta v = |v|^{p_\varepsilon-1}v \text{ in } \mathcal{A}, \quad u = 0 \text{ on } \partial\mathcal{A}, \quad (2)$$

with $p_\varepsilon = \frac{(m+1)+2}{(m+1)-2} - \varepsilon$ has:

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- i) a positive solution v_ε which concentrates and blows-up on a $(m-1)$ -dimensional sphere $\Gamma \subset (\partial\mathcal{A})_a$ as $\varepsilon \rightarrow 0$,
- ii) a positive solution v_ε which concentrates and blows-up on two $(m-1)$ -dimensional orthogonal spheres $\Gamma_1 \subset (\partial\mathcal{A})_a$ and $\Gamma_2 \subset (\partial\mathcal{A})_a$ as $\varepsilon \rightarrow 0$,

Theorem 1.2. *[Case of sign changing solutions] Let $\mathcal{A} \subset \mathbb{R}^{2m}$, $m \geq 2$ and define $(\partial\mathcal{A})_a := \{y \in \partial\mathcal{A} : |y| = a\}$. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, the supercritical problem (2) with $p_\epsilon = \frac{(m+1)+2}{(m+1)-2} - \epsilon$ has:*

- i) a sign changing solution v_ε such that v_ε^+ and v_ε^- concentrate and blow-up on two $(m-1)$ -dimensional orthogonal spheres $\Gamma_+ \subset (\partial\mathcal{A})_a$ and $\Gamma_- \subset (\partial\mathcal{A})_a$, respectively, as $\varepsilon \rightarrow 0$,
- ii) a sign changing solution v_ε such that v_ε^+ and v_ε^- concentrate and blow-up on the same $(m-1)$ -dimensional sphere $\Gamma \subset (\partial\mathcal{A})_a$, as $\varepsilon \rightarrow 0$,
- iii) two sign changing solutions v_ε^1 and v_ε^2 each one is such that $(v_\varepsilon^i)^+$ and $(v_\varepsilon^i)^-$ concentrate and blow-up on two $(m-1)$ -dimensional orthogonal spheres $(\Gamma_i)_+ \subset (\partial\mathcal{A})_a$ and $(\Gamma_i)_- \subset (\partial\mathcal{A})_a$, respectively, as $\varepsilon \rightarrow 0$, $i = 1, 2$.

We remark that the exponent $\frac{(m+1)+2}{(m+1)-2} - \epsilon$ which is almost critical in dimension $(m+1)$ is obviously supercritical for problem (2).

To prove our results we use the reduction method introduced in [14] which allows to transform symmetric solutions to (2) into symmetric solutions of a similar nonhomogeneous problem in an annulus $\mathcal{D} \subset \mathbb{R}^{m+1}$. This method was inspired by the paper [18] where a reduction approach was used to pass from a singularly perturbed problem in an annulus in \mathbb{R}^4 to a singularly perturbed problem in an annulus in \mathbb{R}^3 .

More precisely let us consider the annulus $\mathcal{D} \subset \mathbb{R}^{m+1}$ $\mathcal{D} := \{x \in \mathbb{R}^{m+1} : a^2/2 < |y| < b^2/2\}$, and, write a point $y \in \mathbb{R}^{2m}$ as $y = (y_1, y_2)$, $y_i \in \mathbb{R}^m$, $i = 1, 2$. Then we consider functions v in $\mathcal{A} \subset \mathbb{R}^{2m}$ which are radially symmetric in y_1 and y_2 , i.e. $v(y) = w(|y_1|, |y_2|)$ and functions u in $\mathcal{D} \subset \mathbb{R}^{m+1}$ which are radially symmetric about the x_{m+1} -axis, i.e. $u(x) = h(|x|, \varphi)$ with $\varphi = \arccos\left(\frac{x}{|x|, \underline{\mathcal{E}}_{m+1}}\right)$ where $\underline{\mathcal{E}}_{m+1} = (0, \dots, 0, 1)$. We also set

$$X = \{v \in C^{2,\alpha}(\overline{\mathcal{A}}) : v \text{ is radially symmetric}\}$$

$$Y = \{u \in C^{2,\alpha}(\overline{\mathcal{D}}) : u \text{ is axially symmetric}\}.$$

Then, as corollary of Theorem 1.1 of [14] we have

Proposition 1.3. *There is a bijective correspondence h between solutions v of (2) in X and solutions $u = h(v)$ in Y of the following reduced problem*

$$-\Delta u = \frac{1}{2|x|}|u|^{p_\epsilon-1}u \quad \text{in } \mathcal{D} \subset \mathbb{R}^{m+1}, \quad u = 0 \quad \text{on } \partial\mathcal{D}. \quad (3)$$

As a consequence of this result we can obtain solutions of problem (2) by constructing axially symmetric solutions of the lower-dimensional problem (3). This has the immediate advantage of transforming the supercritical problem (2) into the subcritical problem (3) if the exponent p_ϵ is taken as $\frac{(m+1)+2}{(m+1)-2} - \epsilon$. Indeed we will prove Theorem 1.1 and Theorem 1.2 by constructing axially symmetric solutions of (2.3), positive or sign changing, which blow-up and concentrate in points of the annulus $\mathcal{D} \subset \mathbb{R}^{m+1}$. These solutions will give rise to solutions of (2) concentrating on $(m-1)$ -dimensional spheres, because, as a consequence of the proof of Theorem 1.1 of [14] and of Remark 3.1 of [14] it holds

Proposition 1.4. *If u_ε is an axially symmetric solution of (2) concentrating, as $\varepsilon \rightarrow 0$, on a point ξ which belongs to the $x_{(m+1)}$ -axis, i.e. $\xi = (0, \dots, 0, t)$ for $t \in \mathbb{R} \setminus \{0\}$, then the corresponding solution $v_\varepsilon = h^{-1}(u_\varepsilon)$ concentrates on a $(m-1)$ -dimensional sphere in \mathbb{R}^{2m} .*

This is because, by symmetry considerations and by the change of variable performed in [14] to prove Theorem 1.1 any point ξ on the $x_{(m+1)}$ -axis in $\mathcal{D} \subset \mathbb{R}^{m+1}$ is mapped into a $(m-1)$ -dimensional sphere in $\mathcal{A} \subset \mathbb{R}^{2m}$. We refer to [14] for all details.

Thus let $\Omega := \{x \in \mathbb{R}^n : 1 < |x| < r\}$ be an annulus in \mathbb{R}^n , $n \geq 3$, and consider the problem

$$-\Delta u = \frac{1}{2|x|} |u|^{p-1-\varepsilon} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (4)$$

where $p = \frac{n+2}{n-2}$ and ε is a small positive parameter. Let $U_{\delta, \xi}(x) := \alpha_n \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x-\xi|^2)^{\frac{n-2}{2}}}$ with $\delta > 0$ and $x, \xi \in \mathbb{R}^n$, be the solutions to the critical problem $-\Delta u = u^p$ in \mathbb{R}^n . Here $\alpha_n := [n(n-2)]^{\frac{n-2}{4}}$. We have

Theorem 1.5. *There exists $\epsilon_0 > 0$ such that, for each $\epsilon \in (0, \epsilon_0)$, problem (4) has*

- (i) *an axially symmetric positive solution u_ϵ with one simple positive blow-up point which converge to ξ_0 as ε goes to zero, i.e.*

$$u_\epsilon(x) = U_{\delta_\epsilon, \xi_\epsilon}(x) + o(1) \quad \text{in } H_0^1(\Omega),$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_\epsilon \rightarrow d > 0, \quad \xi_\epsilon \rightarrow \xi_0;$$

- (ii) *an axially symmetric positive solution u_ϵ with two simple positive blow-up points which converge to ξ_0 and $-\xi_0$ as ε goes to zero, i.e.*

$$u_\epsilon(x) = U_{\delta_\epsilon, \xi_\epsilon}(x) + U_{\delta_\epsilon, -\xi_\epsilon}(x) + o(1),$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_\epsilon \rightarrow d > 0, \quad \xi_\epsilon \rightarrow \xi_0;$$

- (iii) *an axially symmetric sign-changing solutions solution u_ϵ with one simple positive and one simple negative blow-up points which converge to ξ_0 and $-\xi_0$ as ε goes to zero, i.e.*

$$u_\epsilon(x) = U_{\delta_\epsilon, \xi_\epsilon}(x) - U_{\delta_\epsilon, -\xi_\epsilon}(x) + o(1),$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_\epsilon \rightarrow d > 0, \quad \xi_\epsilon \rightarrow \xi_0;$$

- (iv) *an axially symmetric sign-changing solutions solution u_ϵ with one double nodal blow-up point which converge to ξ_0 as ε goes to zero, i.e.*

$$u_\epsilon(x) = U_{\delta_{1\epsilon}, \xi_{1\epsilon}}(x) - U_{\delta_{2\epsilon}, \xi_{2\epsilon}}(x) + o(1),$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_{i\epsilon} \rightarrow d_i > 0, \quad \xi_{i\epsilon} \rightarrow \xi_0$$

for $i = 1, 2$.

- (v) *two axially symmetric sign-changing solutions solution u_ϵ with two double nodal blow-up points which converge to ξ_0 and $-\xi_0$ as ϵ goes to zero, i.e.*

$$u_\epsilon(x) = [U_{\delta_{1\epsilon}, \xi_{1\epsilon}}(x) - U_{\delta_{2\epsilon}, \xi_{2\epsilon}}(x)] + [U_{-\delta_{1\epsilon}, -\xi_{1\epsilon}}(x) - U_{-\delta_{2\epsilon}, -\xi_{2\epsilon}}(x)] + o(1)$$

and

$$u_\epsilon(x) = [U_{\delta_{1\epsilon}, \xi_{1\epsilon}}(x) - U_{\delta_{2\epsilon}, \xi_{2\epsilon}}(x)] - [U_{-\delta_{1\epsilon}, -\xi_{1\epsilon}}(x) - U_{-\delta_{2\epsilon}, -\xi_{2\epsilon}}(x)] + o(1)$$

with

$$\epsilon^{-\frac{n-1}{n-2}} \delta_{i\epsilon} \rightarrow d_i > 0, \quad \xi_{i\epsilon} \rightarrow \xi_0$$

for $i = 1, 2$.

Obviously Theorem 1.1 and Theorem 1.2 derive from Theorem 1.5 for $n = m + 1$ using Proposition 1.3 and Proposition 1.4.

The proof of Theorem 1.5 relies on a very well known Ljapunov-Schmidt finite dimensional reduction. We omit many details on the finite dimensional reduction because they can be found, up to some minor modifications, in the literature. In Section 2 we write the approximate solution, we sketch the proof of the Ljapunov-Schmidt procedure and we prove Theorem 1.5. In Section 3 we only compute the expansion of the reduced energy, which is crucial in this framework. In the Appendix we recall some well known facts.

2. THE LJAPUNOV-SCHMIDT PROCEDURE

We equip $H_0^1(\Omega)$ with the inner product $(u, v) = \int_{\Omega} \nabla u \nabla v dx$ and the corresponding norm $\|u\|^2 = \int_{\Omega} |\nabla u|^2 dx$. For $r \in [1, \infty)$ and $u \in L^r(\Omega)$ we set $\|u\|_r^r = \int_{\Omega} |u|^r dx$.

Let us rewrite problem (4) in a different way. Let $i^* : L^{\frac{2n}{n-2}}(\Omega) \rightarrow H_0^1(\Omega)$ be the adjoint operator of the embedding $i : H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, i.e.

$$i^*(u) = v \quad \Leftrightarrow \quad (v, \varphi) = \int_{\Omega} u(x) \varphi(x) dx \quad \forall \varphi \in H_0^1(\Omega).$$

It is clear that there exists a positive constant c such that

$$\|i^*(u)\| \leq c \|u\|^{\frac{2n}{n+2}} \quad \forall u \in L^{\frac{2n}{n+2}}(\Omega).$$

Setting $f_\epsilon(s) := |s|^{p-1-\epsilon} s$ and using the operator i^* , problem (4) turns out to be equivalent to

$$u = i^* \left[\frac{1}{2|x|} f_\epsilon(u) \right], \quad u \in H_0^1(\Omega). \quad (5)$$

Let $U_{\delta, \xi} := \alpha_n \frac{\delta^{\frac{n-2}{2}}}{(\delta^2 + |x - \xi|^2)^{\frac{n-2}{2}}}$, with $\alpha_n := [n(n-2)]^{\frac{n-2}{4}}$ be the positive solutions to the limit problem

$$-\Delta u = u^p \text{ in } \mathbb{R}^n.$$

Set

$$\psi_{\delta, \xi}^0(x) := \frac{\partial U_{\delta, \xi}}{\partial \delta} = \alpha_n \frac{n-2}{2} \delta^{\frac{n-4}{2}} \frac{|x - \xi|^2 - \delta^2}{(\delta^2 + |x - \xi|^2)^{n/2}}$$

and for any $j = 1, \dots, n$

$$\psi_{\delta, \xi}^j(x) := \frac{\partial U_{\delta, \xi}}{\partial \xi_j} = \alpha_n(n-2)\delta^{\frac{n-2}{2}} \frac{x_j - \xi_j}{(\delta^2 + |x - \xi|^2)^{n/2}}.$$

It is well known that the space spanned by the $(n+1)$ functions $\psi_{\delta, \xi}^j$ is the set of the solution to the linearized problem

$$-\Delta \psi = pU_{\delta, \xi}^{p-1} \psi \text{ in } \mathbb{R}^n.$$

We also denote by PW the projection onto $H_0^1(\Omega)$ of a function $W \in D^{1,2}(\mathbb{R}^n)$, i.e.

$$\Delta PW = \Delta W \text{ in } \Omega, \quad PW = 0 \text{ on } \partial\Omega.$$

Set $\xi_0 := (0, \dots, 0, 1)$. We look for two different types of solutions to problem (5). The solutions of the type (i), (ii) and (iii) of Theorem 1.5 will be of the form

$$u_\varepsilon = PU_{\delta, \xi} + \lambda PU_{\mu, \eta} + \phi \quad (6)$$

where $\lambda \in \{-1, 0, +1\}$ ($\lambda = 0$ in case (i), $\lambda = +1$ in case (ii) and $\lambda = -1$ in case (iii)) and the concentration parameters are

$$\mu = \delta \quad \text{and} \quad \delta := \varepsilon^{\frac{n-1}{n-2}} d \text{ for some } d > 0 \quad (7)$$

while the concentration points satisfy

$$\eta = -\xi \quad \text{and} \quad \xi = (1 + \tau)\xi_0, \text{ with } \tau := \varepsilon t, \quad t > 0. \quad (8)$$

On the other hand, the solutions of the type (iv) and (v) of Theorem 1.5 will be of the form

$$u_\varepsilon = PU_{\delta_1, \xi_1} - PU_{\delta_2, \xi_2} + \lambda(PU_{\mu_1, \eta_1} - PU_{\mu_2, \eta_2}) + \phi, \quad (9)$$

where $\lambda \in \{-1, 0, +1\}$ ($\lambda = 0$ in case (iv), $\lambda = +1$ in the first case (v) and $\lambda = -1$ in the second case (v)) and the concentration parameters are

$$\mu_i = \delta_i \quad \text{and} \quad \delta_i := \varepsilon^{\frac{n-1}{n-2}} d_i \quad \text{with} \quad d_i > 0 \quad (10)$$

while the concentration points are aligned along the x_n -axes and satisfy

$$\eta_i = -\xi_i \quad \text{and} \quad \xi_i = (1 + \tau_i)\xi_0 \text{ with } \tau_i := \varepsilon t_i, \quad t_i > 0. \quad (11)$$

Next, we introduce the configuration space Λ where the concentration parameters and the concentration points lie. For solutions of type (6) we set $\mathbf{d} = d \in (0, +\infty)$ and $\mathbf{t} = t \in (0, +\infty)$ and so

$$\Lambda := \{(\mathbf{d}, \mathbf{t}) \in (0, +\infty) \times (0, +\infty)\},$$

while for solutions of type (9) we set $\mathbf{d} = (d_1, d_2) \in (0, +\infty)^2$ and $\mathbf{t} = (t_1, t_2) \in (0, +\infty)^2$ and so

$$\Lambda := \{(\mathbf{d}, \mathbf{t}) \in (0, +\infty)^4 : t_1 < t_2\}.$$

In each of these cases we write

$$V_{\mathbf{d}, \mathbf{t}} := PU_{\delta, \xi} + \lambda PU_{\mu, \eta} \quad \text{or} \quad V_{\mathbf{d}, \mathbf{t}} := PU_{\delta_1, \xi_1} - PU_{\delta_2, \xi_2} + \lambda(PU_{\mu_1, \eta_1} - PU_{\mu_2, \eta_2}).$$

The remainder term ϕ in both cases (6) and (9) belongs to a suitable space which we now define.

We introduce the spaces

$$K_{\mathbf{d}, \mathbf{t}} := \text{span}\{P\psi_{\delta_i, \xi_i}^j, P\psi_{\mu_\kappa, \xi_\kappa}^\ell : i, \kappa = 1, 2, j, \ell = 0, 1, \dots, n\}$$

(we agree that if $\lambda = 0$ then $K_{\mathbf{d},\mathbf{t}}$ is only generated by the $P\psi_{\delta_i, \xi_i}^j$'s) and

$$K_{\mathbf{d},\mathbf{t}}^\perp := \{\phi \in \mathcal{H}_\lambda : (\phi, \psi) = 0 \quad \forall \psi \in K_{\mathbf{d},\mathbf{t}}\},$$

where the space \mathcal{H}_λ depends on $\lambda \in \{-1, 0, +1\}$ and is defined by

$$\mathcal{H}_0 := \{\phi \in H_0^1(\Omega) : \phi \text{ is axially symmetric with respect to the } x_n\text{-axes}\},$$

$$\mathcal{H}_{+1} := \{\phi \in \mathcal{H}_0 : \phi(x_1, \dots, x_n) = \phi(x_1, \dots, -x_n)\},$$

$$\mathcal{H}_{-1} := \{\phi \in \mathcal{H}_0 : \phi(x_1, \dots, x_n) = -\phi(x_1, \dots, -x_n)\}.$$

Then we introduce the orthogonal projection operators $\Pi_{\mathbf{d},\mathbf{t}}$ and $\Pi_{\mathbf{d},\mathbf{t}}^\perp$ in $H_0^1(\Omega)$, respectively.

As usual for this reduction method, the approach to solve problem (4) or (5) will be to find a pair (\mathbf{d}, \mathbf{t}) and a function $\phi \in K_{\mathbf{d},\mathbf{t}}^\perp$ such that

$$\Pi_{\mathbf{d},\mathbf{t}}^\perp \left\{ V_{\mathbf{d},\mathbf{t}} + \phi - i^* \left[\frac{1}{2|x|} f_\varepsilon(V_{\mathbf{d},\mathbf{t}} + \phi) \right] \right\} = 0 \quad (12)$$

and

$$\Pi_{\mathbf{d},\mathbf{t}} \left\{ V_{\mathbf{d},\mathbf{t}} + \phi - i^* \left[\frac{1}{2|x|} f_\varepsilon(V_{\mathbf{d},\mathbf{t}} + \phi) \right] \right\} = 0 \quad (13)$$

First, we shall find for any (\mathbf{d}, \mathbf{t}) and for small ε , a function $\phi \in K_{\mathbf{d},\mathbf{t}}^\perp$ such that (12) holds. To this aim we define a linear operator $L_{\mathbf{d},\mathbf{t}} : K_{\mathbf{d},\mathbf{t}}^\perp \rightarrow K_{\mathbf{d},\mathbf{t}}^\perp$ by

$$L_{\mathbf{d},\mathbf{t}}\phi := \phi - \Pi_{\mathbf{d},\mathbf{t}}^\perp i^* [f'_0(V_{\mathbf{d},\mathbf{t}})\phi].$$

Proposition 2.1. *For any compact sets \mathbf{C} in Λ there exists $\varepsilon_0, c > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any $(\mathbf{d}, \mathbf{t}) \in \mathbf{C}$ the operator $L_{\mathbf{d},\mathbf{t}}$ is invertible and*

$$\|L_{\mathbf{d},\mathbf{t}}\phi\| \geq c\|\phi\| \quad \forall \phi \in K_{\mathbf{d},\mathbf{t}}^\perp.$$

Proof. We argue as in Lemma 1.7 of [12]. □

Now, we are in position to solve equation (12).

Proposition 2.2. *For any compact sets \mathbf{C} in Λ there exists $\varepsilon_0, c, \sigma > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any $(\mathbf{d}, \mathbf{t}) \in \mathbf{C}$ there exists a unique $\phi_{\mathbf{d},\mathbf{t}}^\varepsilon \in K_{\mathbf{d},\mathbf{t}}^\perp$ such that*

$$\Pi_{\mathbf{d},\mathbf{t}}^\perp \left\{ V_{\mathbf{d},\mathbf{t}} + \phi_{\mathbf{d},\mathbf{t}}^\varepsilon - i^* \left[\frac{1}{2|x|} f_\varepsilon(V_{\mathbf{d},\mathbf{t}} + \phi_{\mathbf{d},\mathbf{t}}^\varepsilon) \right] \right\} = 0.$$

Moreover

$$\|\phi_{\mathbf{d},\mathbf{t}}^\varepsilon\| \leq c\varepsilon^{\frac{1}{2}+\sigma}.$$

Proof. First, we estimate the rate of the error term

$$R_{\mathbf{d},\mathbf{t}} := \Pi_{\mathbf{d},\mathbf{t}}^\perp \left\{ V_{\mathbf{d},\mathbf{t}} - i^* \left[\frac{1}{|x|} f_\varepsilon(V_{\mathbf{d},\mathbf{t}}) \right] \right\}$$

as

$$\|R_{\mathbf{d},\mathbf{t}}\|_{\frac{2n}{n+2}} = O\left(\varepsilon^{\frac{1}{2}+\sigma}\right)$$

for some $\sigma > 0$. We argue as in Appendix B of [1] using estimates of Section 3. Then we argue exactly as in Proposition 2.3 of [5]. \square

Now, we introduce the energy functional $J_\varepsilon : H_0^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_\varepsilon(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{p+1-\varepsilon} \int_{\Omega} \frac{1}{2|x|} |u|^{p+1-\varepsilon} dx,$$

whose critical points are the solutions to problem (4). Let us define the reduced energy $\tilde{J}_\varepsilon : \Lambda \rightarrow \mathbb{R}$ by

$$\tilde{J}_\varepsilon(\mathbf{d}, \mathbf{t}) = J_\varepsilon(V_{\mathbf{d}, \mathbf{t}} + \phi_{\mathbf{d}, \mathbf{t}}^\varepsilon).$$

Next, we prove that the critical points of \tilde{J}_ε are the solution to problem (13).

Proposition 2.3. *The function $V_{\mathbf{d}, \mathbf{t}} + \phi_{\mathbf{d}, \mathbf{t}}^\varepsilon$ is a critical point of the functional J_ε if and only if the point (\mathbf{d}, \mathbf{t}) is a critical point of the function \tilde{J}_ε .*

Proof. We argue as in Proposition 1 of [3]. \square

The problem is thus reduced to the search for critical points of \tilde{J}_ε , so it is necessary to compute the asymptotic expansion of \tilde{J}_ε .

Proposition 2.4. *It holds true that*

$$\tilde{J}_\varepsilon(\mathbf{d}, \mathbf{t}) = c_1 + c_2\varepsilon + c_3\varepsilon \log \varepsilon + \varepsilon(1 + |\lambda|)\Phi(\mathbf{d}, \mathbf{t}) + o(\varepsilon),$$

C^0 -uniformly on compact sets of Λ , where

(i) in case (6)

$$\Phi(\mathbf{d}, \mathbf{t}) := c_4 \left(\frac{d}{2t} \right)^{n-2} + c_5 t - c_6 \ln d$$

(ii) in case (9)

$$\begin{aligned} \Phi(\mathbf{d}, \mathbf{t}) := & c_4 \left[\left(\frac{d_1}{2t_1} \right)^{n-2} + \left(\frac{d_2}{2t_2} \right)^{n-2} + 2(d_1 d_2)^{\frac{n-2}{2}} \left(\frac{1}{|t_1 - t_2|^{n-2}} - \frac{1}{|t_1 + t_2|^{n-2}} \right) \right] \\ & + c_5(t_1 + t_2) - c_6(\ln d_1 + \ln d_2). \end{aligned}$$

Here c_i 's are constants and c_4 , c_5 and c_6 are positive.

Proof. The proof is postponed to Section 3. \square

Proof of Theorem 1.5. It is easy to verify that the function Φ of Proposition 2.4 in both cases has a minimum point which is stable under uniform perturbations. Therefore, if ε is small enough there exists a critical point $(\mathbf{d}_\varepsilon, \mathbf{t}_\varepsilon)$ of the reduced energy \tilde{J}_ε . Finally, the claim follows by Proposition 2.3. \square

3. EXPANSION OF THE REDUCED ENERGY

It is standard to prove that

$$\widetilde{J}_\varepsilon(\mathbf{d}, \mathbf{t}) = J_\varepsilon(V_{\mathbf{d}, \mathbf{t}}) + o(\varepsilon)$$

(see for example [3, 5]). So the problem reduces to estimating the leading term $J_\varepsilon(V_{\mathbf{d}, \mathbf{t}})$. We will estimate it in case (9) with $|\lambda| = 1$, because in the other cases its expansion is easier and can be deduced from that. Proposition 2.4 will follow from Lemma 3.1, Lemma 3.2 and Lemma 3.3.

For future reference we define the constants

$$\gamma_1 = \alpha_n^{p+1} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^n} dy, \quad (14)$$

$$\gamma_2 = \alpha_n^{p+1} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} dy, \quad (15)$$

$$\gamma_3 = \alpha_n^{p+1} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^n} \log \frac{1}{(1 + |y|^2)^{\frac{n-2}{2}}} dy. \quad (16)$$

For sake of simplicity, we set $U_i := U_{\delta_i, \xi_i}$ and $V_i := V_{\mu_i, \eta_i}$.

Lemma 3.1. *It holds true that*

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla V_{\mathbf{d}, \mathbf{t}}|^2 dx &= 2\gamma_1 \\ &- \gamma_2 \varepsilon \left[\left(\frac{d_1}{2t_1} \right)^{n-2} + \left(\frac{d_2}{2t_2} \right)^{n-2} + (d_1 d_2)^{\frac{n-2}{2}} \left(\frac{1}{|t_1 - t_2|^{n-2}} - \frac{1}{|t_1 + t_2|^{n-2}} \right) \right] + o(\varepsilon). \end{aligned}$$

Proof. We have

$$\begin{aligned} \int_{\Omega} |\nabla V_{\mathbf{d}, \mathbf{t}}|^2 dx &= \int_{\Omega} |\nabla P U_1|^2 dx + \int_{\Omega} |\nabla P U_2|^2 dx - 2 \int_{\Omega} \nabla P U_1 \nabla P U_2 dx \\ &+ \int_{\Omega} |\nabla P V_1|^2 dx + \int_{\Omega} |\nabla P V_2|^2 dx - 2 \int_{\Omega} \nabla P V_1 \nabla P V_2 dx \\ &+ 2 \sum_{i,j=1}^2 \lambda \int_{\Omega} \nabla P U_i \nabla P V_j dx \\ &= 2 \left(\int_{\Omega} |\nabla P U_1|^2 dx + \int_{\Omega} |\nabla P U_2|^2 dx - 2 \int_{\Omega} \nabla P U_1 \nabla P U_2 dx \right) + o(\varepsilon), \end{aligned} \quad (17)$$

because of the symmetry (see (10) and (11)) and the fact that

$$\int_{\Omega} \nabla P U_i \nabla P V_j dx = O \left(\delta_i^{\frac{n-2}{2}} \mu_j^{\frac{n-2}{2}} \right) = o(\varepsilon).$$

Let us estimate the first term in (17). The estimate of the second term is similar. We set

$$\tau := \min \left\{ d(\xi_1, \partial\Omega), d(\xi_2, \partial\Omega), \frac{|\xi_1 - \xi_2|}{2} \right\} = \min \left\{ \tau_1, \tau_2, \frac{|\tau_1 - \tau_2|}{2} \right\}. \quad (18)$$

We get

$$\int_{\Omega} |\nabla PU_1|^2 dx = \int_{\Omega} U_1^p PU_1 dx = \int_{B(\xi_1, \tau)} U_1^p PU_1 dx + \int_{\Omega \setminus B(\xi_1, \tau)} U_1^p PU_1 dx.$$

By Lemma A.1 we deduce

$$\int_{\Omega \setminus B(\xi_1, \tau)} U_1^p PU_1 dx = O \left(\left(\frac{\delta_1}{\tau} \right)^n \right) = o(\varepsilon)$$

$$\int_{B(\xi_1, \tau)} U_1^p PU_1 dx = \int_{B(\xi_1, \tau)} U_1^{p+1} dx + \int_{B(\xi_1, \tau)} U_1^p (PU_1 - U_1) dx, \quad (19)$$

with

$$\int_{B(\xi_1, \tau)} U_1^{p+1} dx = \gamma_1 + O \left(\left(\frac{\delta_1}{\tau_1} \right)^n \right) = \gamma_1 + o(\varepsilon).$$

The second term in (19) is estimated in (i) of Lemma 3.4.

It remains only to estimate the third term in (17).

$$\int_{\Omega} \nabla PU_1 \nabla PU_2 dx = \int_{\Omega} U_1^p PU_2 dx = \int_{B(\xi_1, \tau)} U_1^p PU_2 dx + \int_{\Omega \setminus B(\xi_1, \tau)} U_1^p PU_2 dx. \quad (20)$$

We have

$$\begin{aligned} \int_{\Omega \setminus B(\xi_1, \tau)} U_1^p PU_2 dx &= O \left(\delta_1^{\frac{n+2}{2}} \delta_2^{\frac{n-2}{2}} \int_{\Omega \setminus B(\xi_1, \tau)} \frac{1}{|x - \xi_1|^{n+2}} \frac{1}{|x - \xi_2|^{n-2}} dx \right) \\ &= O \left(\frac{\delta_1^{\frac{n+2}{2}} \delta_2^{\frac{n-2}{2}}}{\tau^n} \int_{\mathbb{R}^n \setminus B(0, 1)} \frac{1}{|y|^{n+2}} \frac{1}{|y + \frac{\xi_1 - \xi_2}{\tau}|^{n-2}} dy \right) = O \left(\frac{\delta_1^{\frac{n+2}{2}} \delta_2^{\frac{n-2}{2}}}{\tau^n} \right) = o(\varepsilon). \end{aligned}$$

The first term in (20) is estimated in (ii) of Lemma 3.4.

The claim then follows collecting all the previous estimates and taking into account the choice of δ'_i s and τ'_i s made in (6) and (7).

□

Lemma 3.2. *It holds true that*

$$\begin{aligned} \frac{1}{p+1} \int_{\Omega} \frac{1}{|x|} |V_{\mathbf{d}, \mathbf{t}}|^{p+1} dx &= 2 \left[\frac{2}{p+1} \gamma_1 - \frac{1}{p+1} \gamma_1 \varepsilon (t_1 + t_2) \right] \\ &- 2\gamma_2 \varepsilon \left[\left(\frac{d_1}{2t_1} \right)^{n-2} + \left(\frac{d_2}{2t_2} \right)^{n-2} + 2(d_1 d_2)^{\frac{n-2}{2}} \left(\frac{1}{|t_1 - t_2|^{n-2}} - \frac{1}{|t_1 + t_2|^{n-2}} \right) \right] + o(\varepsilon). \end{aligned}$$

Proof. We have

$$\begin{aligned} \int_{\Omega} \frac{1}{|x|} |V_{\mathbf{d}, \mathbf{t}}|^{p+1} dx &= \int_{\Omega} \frac{1}{|x|} |PU_1 - PU_2 + \lambda(PV_1 - PV_2)|^{p+1} dx \\ &= \int_{\Omega} \frac{1}{|x|} (|PU_1 - PU_2 + \lambda(PV_1 - PV_2)|^{p+1} - |U_1|^{p+1} - |U_2|^{p+1} - |V_1|^{p+1} - |V_2|^{p+1}) dx \\ &+ \int_{\Omega} \frac{1}{|x|} (|U_1|^{p+1} + |U_2|^{p+1} + |V_1|^{p+1} + |V_2|^{p+1}) dx \\ &= \int_{\Omega} \frac{1}{|x|} (|PU_1 - PU_2 + \lambda(PV_1 - PV_2)|^{p+1} - |U_1|^{p+1} - |U_2|^{p+1} - |V_1|^{p+1} - |V_2|^{p+1}) dx \\ &+ 2 \int_{\Omega} \frac{1}{|x|} (|U_1|^{p+1} + |U_2|^{p+1}) dx, \end{aligned} \tag{21}$$

because of the symmetry (see (10) and (11)).

The last two terms in (21) are estimated in (v) of Lemma 3.4. Let τ as in (18).

We split the first integral as

$$\begin{aligned} &\int_{\Omega} \frac{1}{|x|} (|PU_1 - PU_2 + \lambda(PV_1 - PV_2)|^{p+1} - |U_1|^{p+1} - |U_2|^{p+1} - |V_1|^{p+1} - |V_2|^{p+1}) dx \\ &= \int_{B(\xi_1, \tau)} \dots + \int_{B(\xi_2, \tau)} \dots + \int_{B(-\xi_1, \tau)} \dots + \int_{B(-\xi_2, \tau)} \dots \\ &+ \int_{\Omega \setminus (B(\xi_1, \tau) \cup B(\xi_2, \tau) \cup B(-\xi_1, \tau) \cup B(-\xi_2, \tau))} \dots \end{aligned} \tag{22}$$

By Lemma A.1 we deduce

$$\begin{aligned}
& \int_{\Omega \setminus (B(\xi_1, \tau) \cup B(\xi_2, \tau) \cup B(-\xi_1, \tau) \cup B(-\xi_2, \tau))} \dots \\
&= O \left(\int_{\Omega \setminus (B(\xi_1, \tau) \cup B(\xi_2, \tau) \cup B(-\xi_1, \tau) \cup B(-\xi_2, \tau))} \left(U_1^{p+1} + U_2^{p+1} + V_1^{p+1} + V_2^{p+1} \right) dx \right) \\
&= O \left(\frac{\delta_1^n}{\tau^n} + \frac{\delta_2^n}{\tau^n} \right) = o(\varepsilon).
\end{aligned}$$

We now estimate the integral over $B(\xi_1, \tau)$ in (22).

$$\begin{aligned}
& \int_{B(\xi_1, \tau)} \frac{1}{|x|} (|PU_1 - PU_2 + \lambda(PV_1 - PV_2)|^{p+1} - |U_1|^{p+1} - |U_2|^{p+1} - |V_1|^{p+1} - |V_2|^{p+1}) dx \\
&= (p+1) \int_{B(\xi_1, \tau)} \frac{1}{|x|} U_1^p (PU_1 - U_1 - PU_2 + \lambda(PV_1 - PV_2)) dx \\
&+ \frac{p(p+1)}{2} \int_{B(\xi_1, \tau)} \frac{1}{|x|} |U_1 + \theta\rho|^{p-1} \rho^2 dx - \int_{B(\xi_1, \tau)} \frac{1}{|x|} (|U_2|^{p+1} + |V_1|^{p+1} - |V_2|^{p+1}) dx \\
&= (p+1) \int_{B(\xi_1, \tau)} \frac{1}{|x|} U_1^p (PU_1 - U_1) dx - (p+1) \int_{B(\xi_1, \tau)} \frac{1}{|x|} U_1^p PU_2 dx + o(\varepsilon), \tag{23}
\end{aligned}$$

where $\rho := PU_1 - U_1 - PU_2 + \lambda(PV_1 - PV_2)$. Indeed, by Lemma A.1 one can easily deduce that

$$\int_{B(\xi_1, \tau)} \frac{1}{|x|} U_1^p (PV_1 - PV_2) dx, \quad \int_{B(\xi_1, \tau)} \frac{1}{|x|} |U_2|^{p+1} dx, \quad \int_{B(\xi_1, \tau)} \frac{1}{|x|} |V_i|^{p+1} dx = o(\varepsilon)$$

and also

$$\begin{aligned}
& \frac{p(p+1)}{2} \int_{B(\xi_1, \tau)} \frac{1}{|x|} |U_1 + \theta\rho|^{p-1} \rho^2 dx \leq c \int_{B(\xi_1, \tau)} |U_1|^{p-1} \rho^2 dx + \int_{B(\xi_1, \tau)} |\rho|^{p+1} dx \\
&\leq c \int_{B(\xi_1, \tau)} U_1^{p-1} (PU_1 - U_1)^2 dx + c \int_{B(\xi_1, \tau)} U_1^{p-1} (PU_2)^2 dx + c \int_{B(\xi_1, \tau)} U_1^{p-1} (PV_1 - PV_2)^2 dx \\
&+ c \int_{B(\xi_1, \tau)} |PU_1 - U_1|^{p+1} + c \int_{B(\xi_1, \tau)} |U_2|^{p+1} + c \int_{B(\xi_1, \tau)} (|V_1|^{p+1} + |V_2|^{p+1}) dx \\
&= o(\varepsilon).
\end{aligned}$$

The first term and the second term in (23) are estimated in (iii) and (iv) of Lemma 3.4, respectively. Therefore, the claim follows. \square

Lemma 3.3. *It holds true that*

$$\begin{aligned} \frac{1}{p+1-\varepsilon} \int_{\Omega} \frac{1}{|x|} |V_{\mathbf{d},\mathbf{t}}|^{p+1-\varepsilon} &= \frac{1}{p+1} \int_{\Omega} \frac{1}{|x|} |V_{\mathbf{d},\mathbf{t}}|^{p+1} \\ &+ (1+|\lambda|) \left[\frac{\gamma_1}{(p+1)^2} - \alpha_n \frac{\gamma_1}{(p+1)} - \frac{\gamma_3}{(p+1)} \varepsilon + \frac{n-2}{2(p+1)} (\ln \delta_1 + \ln \delta_2) \right] + o(\varepsilon). \end{aligned}$$

Proof. We argue exactly as in Lemma 3.2 of [7]. □

Lemma 3.4. *Let τ as in (18). It holds true that*

(i)

$$\int_{B(\xi_1, \tau)} U_1^p (PU_1 - U_1) dx = -\gamma_2 \left(\frac{\delta_1}{2\tau_1} \right)^{n-2} + o(\varepsilon)$$

(ii)

$$\int_{B(\xi_1, \tau)} U_1^p PU_2 dx = \gamma_2 (\delta_1 \delta_2)^{\frac{n-2}{2}} \left(\frac{1}{|\tau_1 - \tau_2|^{n-2}} - \frac{1}{|\tau_1 + \tau_2|^{n-2}} \right) + o(\varepsilon)$$

(iii)

$$\int_{B(\xi_1, \tau)} \frac{1}{|x|} U_1^p (PU_1 - U_1) dx = -\gamma_2 \left(\frac{\delta_1}{2\tau_1} \right)^{n-2} + o(\varepsilon)$$

(iv)

$$\int_{B(\xi_1, \tau)} \frac{1}{|x|} U_1^p PU_2 dx = -\gamma_2 \left(\frac{\delta_1}{2\tau_1} \right)^{n-2} + o(\varepsilon)$$

(v)

$$\int_{\Omega} \frac{1}{|x|} U_1^{p+1} dx = \gamma_1 - \gamma_1 \tau_1 + o(\varepsilon).$$

Proof. *Proof of (i)* By Lemma A.1 we get

$$\begin{aligned} \int_{B(\xi_1, \tau)} U_1^p (PU_1 - U_1) dx &= \int_{B(\xi_1, \tau)} U_1^p \left(-\alpha_n \delta_1^{\frac{n-2}{2}} H(x, \xi_1) + R_{\delta_1, \xi_1} \right) dx \\ &= -\alpha_n \delta_1^{\frac{n-2}{2}} \int_{B(\xi_1, \tau)} U_1^p H(x, \xi_1) dx + \int_{B(\xi_1, \tau)} U_1^p R_{\delta_1, \xi_1} dx, \end{aligned}$$

with

$$\int_{B(\xi_1, \tau)} U_1^p R_{\delta_1, \xi_1} dx = O \left(\left(\frac{\delta_1}{\tau_1} \right)^n \right).$$

By Lemma 3.5 we get

$$\begin{aligned}
\alpha_n \delta_1^{\frac{n-2}{2}} \int_{B(\xi_1, \tau)} U_1^p H(x, \xi_1) dx &= \alpha_n^{p+1} \delta_1^{n-2} \int_{B(0, \tau/\delta_1)} H(\delta_1 y + \xi_1, \xi_1) \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} dy \\
&= \alpha_n^{p+1} \left(\frac{\delta_1}{\tau_1} \right)^{n-2} \int_{B(0, \tau/\delta_1)} \tau_1^{n-2} H(\delta_1 y + \xi_1, \xi_1) \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} dy \\
&= \alpha_n^{p+1} \left(\frac{\delta_1}{\tau_1} \right)^{n-2} \left[\frac{1}{2^{n-2}} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} dy + o(1) \right].
\end{aligned}$$

Proof of (ii) By Lemma A.1 and Lemma 3.5 we get

$$\begin{aligned}
\int_{B(\xi_1, \tau)} U_1^p P U_2 dx &= \int_{B(\xi_1, \tau)} U_1^p \left(U_2 - \alpha_n \delta_2^{\frac{n-2}{2}} H(x, \xi_2) + R_{\delta_2, \xi_2} \right) dx \\
&= \alpha_n^{p+1} (\delta_1 \delta_2)^{\frac{n-2}{2}} \int_{B(0, \tau/\delta_1)} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} \frac{1}{(\delta_2^2 + |\delta_1 y + \xi_1 - \xi_2|^2)^{\frac{n-2}{2}}} dy \\
&\quad - \alpha_n^{p+1} (\delta_1 \delta_2)^{\frac{n-2}{2}} \int_{B(0, \tau/\delta_1)} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} H(\delta_1 y + \xi_1, \xi_2) dy \\
&\quad + \alpha_n^{p+1} (\delta_1 \delta_2)^{\frac{n-2}{2}} \int_{B(0, \tau/\delta_1)} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} R_{\delta_2, \xi_2}(\delta_1 y + \xi_1) dy = \\
&= \alpha_n^{p+1} \frac{(\delta_1 \delta_2)^{\frac{n-2}{2}}}{|\tau_1 - \tau_2|^{n-2}} \int_{B(0, \tau/\delta_1)} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} \frac{|\tau_1 - \tau_2|^{n-2}}{(\delta_2^2 + |\delta_1 y + \xi_1 - \xi_2|^2)^{\frac{n-2}{2}}} dy \\
&\quad - \alpha_n^{p+1} \frac{(\delta_1 \delta_2)^{\frac{n-2}{2}}}{|\tau_1 + \tau_2|^{n-2}} \int_{B(0, \tau/\delta_1)} \frac{|\tau_1 + \tau_2|^{n-2}}{(1+|y|^2)^{\frac{n+2}{2}}} H(\delta_1 y + \xi_1, \xi_2) dy \\
&\quad + O \left((\delta_1 \delta_2)^{\frac{n-2}{2}} \frac{\delta_2^{\frac{n+2}{2}}}{\tau_2^n} \right) = \\
&= \alpha_n^{p+1} \frac{(\delta_1 \delta_2)^{\frac{n-2}{2}}}{|\tau_1 - \tau_2|^{n-2}} \left[\int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} dy + o(1) \right] \\
&\quad - \alpha_n^{p+1} \frac{(\delta_1 \delta_2)^{\frac{n-2}{2}}}{|\tau_1 + \tau_2|^{n-2}} \left[\int_{\mathbb{R}^n} \frac{1}{(1+|y|^2)^{\frac{n+2}{2}}} dy + o(1) \right] \\
&\quad + o \left(\frac{(\delta_1 \delta_2)^{\frac{n-2}{2}}}{\tau_2^{n-2}} \right).
\end{aligned}$$

Proof of (iii) and (iv) We argue as in the proof of (i) and (ii) using estimates (25) and (26).

Proof of (v) We have

$$\int_{\Omega} \frac{1}{|x|} U_1^{p+1} dx = \int_{B(\xi_1, \tau)} U_1^{p+1} dx + \int_{\Omega \setminus B(\xi_1, \tau)} U_1^{p+1} dx, \tag{24}$$

with

$$\int_{\Omega \setminus B(\xi_1, \tau)} \frac{1}{|x|} U_1^{p+1} dx = O \left(\frac{\delta_1^n}{\tau^n} \right),$$

So, we only have to estimate the first term in (24). We split it as

$$\int_{B(\xi_1, \tau)} \frac{1}{|x|} U_1^{p+1} dx = \int_{B(\xi_1, \tau)} U_1^{p+1} dx + \int_{B(\xi_1, \tau)} \left(\frac{1}{|x|} - 1 \right) U_1^{p+1} dx.$$

We have

$$\int_{B(\xi_1, \tau)} U_1^{p+1} dx = \gamma_1 + O\left(\frac{\delta_1^n}{\tau^n}\right).$$

Since $\xi_1 = \xi_0(1 + \tau_1)$ and $|\xi_0| = 1$, by the mean value theorem we get

$$\frac{1}{|\delta_1 y + \tau_1 \xi_0 + \xi_0|} - 1 = -\tau_1 - \delta_1 \langle y, \xi_0 \rangle + R(y), \quad (25)$$

where R satisfies the uniform estimate

$$|R(y)| \leq c (\delta_1^2 |y|^2 + \delta_1 \tau_1 |y| + \tau_1^2) \text{ for any } y \in B(0, \tau/\delta_1). \quad (26)$$

Therefore we conclude

$$\begin{aligned} \int_{B(\xi_1, \tau)} \left(\frac{1}{|x|} - 1 \right) U_1^{p+1} dx &= \alpha_n^{p+1} \int_{B(0, \tau/\delta_1)} \left(\frac{1}{|\delta_1 y + \tau_1 \xi_0 + \xi_0|} - 1 \right) \frac{1}{(1 + |y|^2)^n} dy \\ &= \alpha_n^{p+1} \int_{B(0, \tau/\delta_1)} (-\tau_1 - \delta_1 \tau_1 \langle y, \xi_0 \rangle + R(y)) \frac{1}{(1 + |y|^2)^n} dy = -\gamma_1 \tau_1 + o(\tau). \end{aligned}$$

Collecting all the previous estimates we get the claim. \square

Lemma 3.5. *Let τ as in (18). It holds true that*

(i)

$$\int_{B(0, \tau/\delta_1)} \tau_1^{n-2} H(\delta_1 y + \xi_1, \xi_1) \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} dy = \frac{1}{2^{n-2}} \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} dy + o(1),$$

(ii)

$$\int_{B(0, \tau/\delta_1)} \frac{|\tau_1 + \tau_2|^{n-2}}{(1 + |y|^2)^{\frac{n+2}{2}}} H(\delta_1 y + \xi_1, \xi_2) dy = \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} dy + o(1),$$

(iii)

$$\int_{B(0, \tau/\delta_1)} \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} \frac{|\tau_1 - \tau_2|^{n-2}}{(\delta_2^2 + |\delta_1 y + \xi_1 - \xi_2|^2)^{\frac{n-2}{2}}} dy = \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} dy + o(1).$$

Proof. We are going to use Lebesgue's dominated convergence Theorem together with Lemma A.2. First of all, taking into account that $\xi_1 = (1 + \tau_1)\xi_0$ and $\xi_1 = (1 - \tau_1)\xi_0$ we deduce that

$$\tau_1^{n-2} H(\delta_1 y + \xi_1, \xi_1) \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} \rightarrow \frac{1}{2^{n-2}} \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} \text{ a.e. in } \mathbb{R}^n$$

and also that

$$H(\delta_1 y + \xi_1, \xi_1) \leq C_2 \frac{1}{|\delta_1 y + \xi_1 - \bar{\xi}_1|^{n-2}} = C_2 \frac{1}{|\delta_1 y + 2\tau_1 \xi_0|^{n-2}} \leq C_2 \frac{1}{\tau_1^{n-2}},$$

since

$$|\delta_1 y + 2\tau_1 \xi_0| \geq 2\tau_1 - |\delta_1 y| \geq \tau_1 \text{ for any } y \in B(0, \tau/\delta_1).$$

That proves (i).

In a similar way, taking into account that $\xi_1 = (1 + \tau_1)\xi_0$ and $\bar{\xi}_2 = (1 - \tau_2)\xi_0$ we get

$$(\tau_2 + \tau_1)^{n-2} H(\delta_1 y + \xi_1, \xi_2) \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} \rightarrow \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} \text{ a.e. in } \mathbb{R}^n$$

and also that

$$H(\delta_1 y + \xi_1, \xi_2) \leq C_2 \frac{1}{|\delta_1 y + \xi_1 - \bar{\xi}_2|^{n-2}} = C_2 \frac{1}{|\delta_1 y + (\tau_1 + \tau_2)\xi_0|^{n-2}} \leq C_2 \frac{1}{\tau_2^{n-2}},$$

since

$$|\delta_1 y + (\tau_1 + \tau_2)\xi_0| \geq \tau_1 + \tau_2 - |\delta_1 y| \geq \tau_2 \text{ for any } y \in B(0, \tau/\delta_1).$$

That proves (ii).

Finally, we have

$$\frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} \frac{|\tau_1 - \tau_2|^{n-2}}{(\delta_2^2 + |\delta_1 y + \xi_1 - \xi_2|^2)^{\frac{n-2}{2}}} \rightarrow \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}} \text{ a.e. in } \mathbb{R}^n$$

and also that

$$\frac{1}{(\delta_2^2 + |\delta_1 y + \xi_1 - \xi_2|^2)^{\frac{n-2}{2}}} \leq \frac{1}{|\delta_1 y + \xi_1 - \xi_2|^{n-2}} \leq \frac{2^{n-2}}{|\tau_1 - \tau_2|^{n-2}}$$

since

$$|\delta_1 y + \xi_1 - \xi_2| \geq |\xi_1 - \xi_2| - |\delta_1 y| \geq \frac{|\xi_1 - \xi_2|}{2} \text{ for any } y \in B(0, \tau/\delta_1).$$

That proves (iii). □

APPENDIX A.

Here we recall some well known facts which are useful to get estimates in Section 3.

We denote by $G(x, y)$ the Green's function associated to $-\Delta$ with Dirichlet boundary condition and $H(x, y)$ its regular part, i.e.

$$-\Delta_x G(x, y) = \delta_y(x) \quad \text{for } x \in \Omega, \quad G(x, y) = 0 \quad \text{for } x \in \partial\Omega,$$

and

$$G(x, y) = \gamma_n \left(\frac{1}{|x - y|^{n-2}} - H(x, y) \right) \quad \text{where} \quad \gamma_n = \frac{1}{(n-2)|S^{n-1}|}$$

($|S^{n-1}| = (2\pi^{n/2})/\Gamma(n/2)$ denotes the Lebesgue measure of the $(n-1)$ -dimensional unit sphere).

The following lemma was proved in [17].

Lemma A.1. *It holds true that*

$$PU_{\delta,\xi}(x) = U_{\delta,\xi}(x) - \alpha_n \delta^{\frac{n-2}{2}} H(x, \xi) + O\left(\frac{\delta^{\frac{n+2}{2}}}{\text{dist}(\xi, \partial\Omega)^n}\right)$$

for any $x \in \Omega$.

Since Ω is smooth, we can choose small $\epsilon > 0$ such that, for every $x \in \Omega$ with $\text{dist}(x, \partial\Omega) \leq \epsilon$, there is a unique point $x_\nu \in \partial\Omega$ satisfying $\text{dist}(x, \partial\Omega) = |x - x_\nu|$. For such $x \in \Omega$, we define $x^* = 2x_\nu - x$ the reflection point of x with respect to $\partial\Omega$.

The following two lemmas are proved in [1].

Lemma A.2. *It holds true that*

$$\left| H(x, y) - \frac{1}{|\bar{x} - y|^{n-2}} \right| = O\left(\frac{\text{dist}(x, \partial\Omega)^n}{|\bar{x} - y|^{n-2}}\right)$$

and

$$\left| \nabla_x \left(H(x, y) - \frac{1}{|\bar{x} - y|^{n-2}} \right) \right| = O\left(\frac{1}{|\bar{x} - y|^{n-2}}\right)$$

for any $x \in \Omega$ with $\text{dist}(x, \partial\Omega) \leq \epsilon$.

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